

# MAXIMUM PRINCIPLE AND CONVERGENCE OF FUNDAMENTAL SOLUTIONS FOR THE RICCI FLOW

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**ABSTRACT.** In this paper we will prove a maximum principle for the solutions of linear parabolic equation on complete non-compact manifolds with a time varying metric. We will prove the convergence of the Neumann Green function of the conjugate heat equation for the Ricci flow in  $B_k \times (0, T)$  to the minimal fundamental solution of the conjugate heat equation as  $k \rightarrow \infty$ . We will prove the uniqueness of the fundamental solution under some exponential decay assumption on the fundamental solution. We will also give a detail proof of the convergence of the fundamental solutions of the conjugate heat equation for a sequence of pointed Ricci flow  $(M_k \times (-\alpha, 0], x_k, g_k)$  to the fundamental solution of the limit manifold as  $k \rightarrow \infty$  which was used without proof by Perelman in his proof of the pseudolocality theorem for Ricci flow [P].

Maximum principle for the heat equation on complete non-compact manifold with a fixed metric was proved by P. Li, L. Karp [LK] and J. Wang [W] (cf. [CLN]). Maximum principle for parabolic equations on complete non-compact manifold with a metric with uniformly bounded Riemannian curvature and evolving by the Ricci flow,

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{0.1}$$

was proved by W.X. Shi [S1], [S2], [S3] under either a uniform boundedness condition on the solution or some structural conditions on the parabolic equation or positivity assumption on the Riemannian curvature operator.

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Let  $M$  be a non-compact manifold with a time varying metric  $g(t) = (g_{ij}(t))$ ,  $0 \leq t < T$ , such that for any  $0 \leq t < T$   $(M, g(t))$  is a complete non-compact manifold. Let  $x_0$  be a fixed point of  $M$ . In this paper we will prove the maximum principle for the subsolution of the linear parabolic equation,

$$u_t = \Delta u + \vec{a} \cdot \nabla u + bu \quad \text{in } M \times (0, T) \quad (0.2)$$

under the condition

$$\int_0^T \int_M u_+^2(x, t) e^{-\lambda r_t(x_0, x)^2} dV_t dt < \infty \quad (0.3)$$

for some constant  $\lambda > 0$ , vector field  $\vec{a}(\cdot, t)$ ,  $0 \leq t \leq T$ , and function  $b(x, t)$  on  $M \times [0, T]$  where  $u_+ = \max(u, 0)$  and  $r_t(x_0, x)$  is the distance between  $x_0$  and  $x$  with respect to the metric  $g_{ij}(t)$ .

In [CTY] A. Chau, L.F. Tam and C. Yu proved the existence of minimal fundamental solution of the conjugate heat equation of Ricci flow on any  $n$ -dimensional non-compact complete manifold,  $n \geq 3$ , by approximating it by a monotone increasing sequence of Dirichlet Green functions of the conjugate heat equation of Ricci flow in bounded domains. In this paper we will show that their argument can be modified to work for any  $n \geq 2$ . We will prove that the Neumann Green functions of the conjugate heat equation of Ricci flow in bounded domains will also converge to the minimal fundamental solution of the conjugate heat equation of [CTY] for any  $n \geq 2$ .

We will prove the uniqueness of the fundamental solution of the conjugate heat equation under some exponential decay assumption on the fundamental solution. We will also give a detail proof of the convergence of the fundamental solutions of the conjugate heat equation for a sequence of pointed Ricci flow  $(M_k \times (-\alpha, 0], x_k, g_k)$  to the fundamental solution of the limit manifold as  $k \rightarrow \infty$  which was used without proof by Perelman in his proof of the pseudolocality theorem for Ricci flow [P].

We start with some definitions. Let  $x_0 \in M$  and let  $r(x, y) = r_0(x, y)$ ,  $r_t(x) = r_t(x_0, x)$ ,  $r(x) = r(x_0, x) = r_0(x_0, x)$ . Let  $\nabla^t$  and  $\Delta^t$  be the covariant derivative and Laplacian with respect to the metric  $g(t)$ . When there is no ambiguity, we will drop the superscript and write  $\nabla$ ,  $\Delta$ , for  $\nabla^t$ ,  $\Delta^t$ , respectively. For any  $R > 0$ ,  $y \in M$ , let  $B_R^t(y) = B_{g(t)}(y, R)$  be the geodesic ball with center  $y$  and radius  $R$  with respect to the metric  $g(t)$  and let  $B_R = B_R^0(x_0)$ . Let  $dV_t$ ,  $dV$ , be the volume element with respect to the metric  $g(t)$  and  $g(0)$  respectively and let  $V_x(r) = \text{Vol}_{g(0)}(B_r(x))$ .

## Section 1

**Theorem 1.1.** *Let  $M$  be a non-compact manifold with a time varying metric  $g(t) = (g_{ij}(t))$ ,  $0 \leq t < T$ , such that for any  $0 \leq t < T$   $(M, g(t))$  is a complete non-compact manifold. Let  $\vec{a}(\cdot, t)$ ,  $0 \leq t < T$ , be a vector field on  $M$  which satisfies*

$$\sup_{M \times [0, T)} |\vec{a}| \leq \alpha_1 \quad (1.1)$$

and let  $b \in L^\infty(M \times [0, T))$  such that

$$\sup_{M \times [0, T)} |b| \leq \alpha_2 \quad (1.2)$$

for some constants  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ . Suppose  $g(t)$  satisfies

$$-\alpha_3 g_{ij} \leq \frac{\partial g_{ij}}{\partial t} \leq \alpha_3 g_{ij} \quad \text{in } M \times (0, T) \quad (1.3)$$

for some constant  $\alpha_3 > 0$  and  $u \in C(M \times [0, T)) \cap C^{2,1}(M \times (0, T))$  is a subsolution of (0.2) satisfying (0.3) for some constant  $\lambda > 0$  and

$$u(x, 0) \leq 0 \quad \forall x \in M. \quad (1.4)$$

Then

$$u(x, t) \leq 0 \quad \text{on } M \times [0, T). \quad (1.5)$$

*Proof.* We will use a modification of the proof in [EH], [LK], [NT] and [W] to prove the theorem. Let  $x_0 \in M$ ,  $r_t(x) = r_t(x_0, x)$ ,  $r(x) = r_0(x)$ , and

$$h(x, t) = -\frac{r(x)^2}{4(2\eta - t)} \quad \forall x \in M, 0 \leq t \leq \eta$$

for some constant  $0 < \eta \leq (\log(9/8))/\alpha_3$  to be determined later. Then  $h$  satisfies

$$h_t + |\nabla^0 h|_{g(0)}^2 = 0 \quad \text{in } M \times [0, \eta]. \quad (1.6)$$

Choose a smooth function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $0 \leq \phi \leq 1$ , such that  $\phi(x) = 1$  for all  $x \leq 0$ ,  $\phi(x) = 0$  for all  $x \geq 1$  and  $-2 \leq \phi'(x) \leq 0$  for any  $x \in \mathbb{R}$ . For any  $R \geq 1$ , let  $\phi_R(x) = \phi(r(x) - R)$ . Then  $|\nabla^0 \phi_R|_{g(0)} \leq 2$  on  $M$ . Now by (1.3),

$$\begin{cases} e^{-\alpha_3 t} g(0) \leq g(t) \leq e^{\alpha_3 t} g(0) & \text{in } M \times [0, T) \\ e^{-\alpha_3 t} g^{-1}(0) \leq g^{-1}(t) \leq e^{\alpha_3 t} g^{-1}(0) & \text{in } M \times [0, T) \end{cases} \quad (1.7)$$

$$\Rightarrow e^{-\alpha_3 T/2} r(x) \leq r_t(x) \leq e^{\alpha_3 T/2} r(x) \quad \forall x \in M, 0 \leq t < T \quad (1.8)$$

and

$$\begin{aligned} \left| \frac{\partial}{\partial t} (dV_t) \right| &\leq \frac{n\alpha_3}{2} dV_t && \text{in } M \quad \forall 0 \leq t < T \\ \Rightarrow e^{-\frac{n\alpha_3}{2} T} V_s &\leq dV_t \leq e^{\frac{n\alpha_3}{2} T} dV_s && \text{in } M \quad \forall 0 \leq s, t < T. \end{aligned} \quad (1.9)$$

Hence  $|\nabla^t \phi_R| \leq 2e^{\alpha_3 T/2}$  on  $M \times [0, T)$ . By (1.6) and (1.7),

$$h_t + e^{-\alpha_3 \eta} |\nabla h|^2 \leq 0 \quad \text{in } M \times [0, \eta]. \quad (1.10)$$

Then by (0.2), (1.9) and (1.10),

$$\begin{aligned}
& \frac{\partial}{\partial t} \left( \int_M \phi_R^2 e^h u_+^2 dV_t \right) \\
&= \int_M \phi_R^2 e^h h_t u_+^2 dV_t + 2 \int_M \phi_R^2 e^h u_+ u_t dV_t + \int_M \phi_R^2 e^h u_+^2 \frac{\partial}{\partial t} (dV_t) \\
&\leq \int_M \phi_R^2 e^h h_t u_+^2 dV_t + 2 \int_M \phi_R^2 e^h u_+ \Delta u dV_t + 2 \int_M \phi_R^2 e^h u_+ \vec{a} \cdot \nabla u dV_t + 2 \int_M \phi_R^2 e^h b u_+^2 dV_t \\
&\quad + \frac{n\alpha_3}{2} \int_M \phi_R^2 e^h u_+^2 dV_t \\
&\leq -e^{-\alpha_3 \eta} \int_M \phi_R^2 e^h |\nabla h|^2 u_+^2 dV_t - 2 \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t - 2 \int_M \phi_R^2 e^h u_+ \nabla h \cdot \nabla u_+ dV_t \\
&\quad - 4 \int_M \phi_R e^h u_+ \nabla \phi_R \cdot \nabla u_+ dV_t + 2\alpha_1 \int_M \phi_R^2 e^h u_+ |\nabla u_+| dV_t \\
&\quad + \left( 2\alpha_2 + \frac{n\alpha_3}{2} \right) \int_M \phi_R^2 e^h u_+^2 dV_t \quad \forall 0 \leq t < \eta. \tag{1.11}
\end{aligned}$$

Now  $\forall 0 \leq t < \eta$ ,

$$\begin{aligned}
2 \left| \int_M \phi_R^2 e^h u_+ \nabla h \cdot \nabla u_+ dV_t \right| &\leq e^{-\alpha_3 \eta} \int_M \phi_R^2 e^h |\nabla h|^2 u_+^2 dV_t + e^{\alpha_3 \eta} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t \\
&\leq e^{-\alpha_3 \eta} \int_M \phi_R^2 e^h |\nabla h|^2 u_+^2 dV_t + \frac{9}{8} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t, \tag{1.12}
\end{aligned}$$

$$\begin{aligned}
& 4 \left| \int_M \phi_R e^h u_+ \nabla \phi_R \cdot \nabla u_+ dV_t \right| \\
&\leq \frac{1}{2} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t + 8 \int_M e^h |\nabla \phi_R|^2 u_+^2 dV_t \\
&\leq \frac{1}{2} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t + 32e^{\alpha_3 T} \int_{B_{R+1} \setminus B_R} e^h u_+^2 dV_t \tag{1.13}
\end{aligned}$$

and

$$2\alpha_1 \left| \int_M \phi_R^2 e^h u_+ |\nabla u_+| dV_t \right| \leq \frac{1}{4} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t + 4\alpha_1^2 \int_M \phi_R^2 e^h u_+^2 dV_t. \tag{1.14}$$

By (1.11), (1.12), (1.13) and (1.14),

$$\begin{aligned}
& \frac{\partial}{\partial t} \left( \int_M \phi_R^2 e^h u_+^2 dV_t \right) \\
& \leq -\frac{1}{8} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t + C_1 \int_M \phi_R^2 e^h u_+^2 dV_t + 32e^{\alpha_3 T} \int_{B_{R+1} \setminus B_R} e^h u_+^2 dV_t \\
& \Rightarrow \frac{\partial}{\partial t} \left( e^{-C_1 t} \int_M \phi_R^2 e^h u_+^2 dV_t \right) + \frac{e^{-C_1 t}}{8} \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t \\
& \leq 32e^{\alpha_3 T} \int_{B_{R+1} \setminus B_R} e^h u_+^2 dV_t \quad \forall 0 \leq t < \eta \\
& \Rightarrow e^{-C_1 t} \int_M \phi_R^2 e^h u_+^2 dV_t + \frac{e^{-C_1 \eta}}{8} \int_0^t \int_M \phi_R^2 e^h |\nabla u_+|^2 dV_t dt \\
& \leq 32e^{\alpha_3 T} \int_0^\eta \int_{B_{R+1} \setminus B_R} e^h u_+^2 dV_t dt \quad \forall 0 \leq t < \eta
\end{aligned} \tag{1.15}$$

where  $C_1 = 2\alpha_2 + 4\alpha_1^2 + (n\alpha_3/2)$ . By (0.3) and (1.8),

$$\int_0^\eta \int_M u_+^2(x, t) e^{-\lambda_1 r(x)^2} dV_t dt < \infty \tag{1.16}$$

where  $\lambda_1 = \lambda e^{\alpha_3 T}$ . We now choose  $\eta = \min(1/(8\lambda_1), (\log(9/8))/\alpha_3)$ . Then

$$h(x, t) \leq -\lambda_1 r(x)^2 \quad \forall x \in M, 0 \leq t < \eta. \tag{1.17}$$

By (1.16) and (1.17),

$$\int_0^\eta \int_M e^h u_+^2(x, t) dV_t dt < \infty. \tag{1.18}$$

Letting  $R \rightarrow \infty$  in (1.15), by (1.18) we get

$$\begin{aligned}
& e^{-C_1 t} \int_M e^h u_+^2 dV_t + \frac{e^{-C_1 \eta}}{8} \int_0^\eta \int_M e^h |\nabla u_+|^2 dV_t dt = 0 \quad \forall x \in M, 0 \leq t < \min(T, \eta) \\
& \Rightarrow u_+(x, t) = 0 \quad \forall x \in M, 0 \leq t < \min(T, \eta).
\end{aligned}$$

If  $T \leq \eta$ , we are done. If  $T > \eta$ , we repeat the above argument a finite number of times and the theorem follows.

**Corollary 1.2.** (Lemma 6.2 of [CTY]) Let  $(M, g(t))$  be a complete solution of the Ricci flow (0.1) in  $(0, T)$  with

$$|Rm| \leq k_0 \quad \text{on } M \times [0, T] \tag{1.19}$$

for some constant  $k_0 > 0$ . Let  $u \in C(M \times [0, T)) \cap C^{2,1}(M \times (0, T))$  satisfy

$$\Delta u \geq u_t \quad \text{in } M \times (0, T)$$

and (0.3), (1.4), for some constant  $\lambda > 0$ . Then  $u$  satisfies (1.5).

**Corollary 1.3.** *Let  $(M, g(t))$  with  $0 \leq t < T$ ,  $\vec{a}$ ,  $b$  and  $\alpha_3$  be as given in Theorem 1.1. Suppose (1.19) holds for some constant  $k_0 > 0$ . Let  $u \in L^\infty(M \times [0, T]) \cap C(M \times [0, T]) \cap C^{2,1}(M \times (0, T))$  be a subsolution of (0.2) in  $M \times (0, T)$  which satisfies (1.4). Then  $u$  satisfies (1.5).*

*Proof.* By the proof of Theorem 1.1  $u$  satisfies (1.8). By the volume comparison theorem [C], there exist constants  $c_1 > 0$ ,  $c_2 > 0$  such that for any  $\lambda > 0$ ,  $R > 0$ ,

$$\int_0^T \int_{B_R} u_+^2(x, t) e^{-\lambda r_t(x_0, x)^2} dV_t dt \leq c_1 \|u\|_\infty^2 \int_0^T \int_0^\infty e^{c_2 r - \lambda e^{-\alpha_3 T} r^2} dr dt < \infty.$$

Letting  $R \rightarrow \infty$ , we get (0.3). Thus the corollary follows from Theorem 1.1.

**Theorem 1.4.** *Let  $M$  be a  $n$ -dimensional non-compact manifold,  $n \geq 2$ , such that  $(M, g(t))$  is a complete solution of the backward Ricci flow*

$$\frac{\partial}{\partial t} g_{ij} = 2R_{ij} \quad (1.20)$$

*in  $[0, T]$  which satisfies (1.19) for some constant  $k_0 > 0$ . Let  $\mathcal{Z}(x, t; y, s)$ ,  $x, y \in M$ ,  $0 \leq s < t \leq T$ , be the minimal fundamental solution of the forward conjugate heat equation in  $M \times (s, T]$ . That is  $\mathcal{Z}(\cdot, \cdot; y, s)$  satisfies (cf. [CTY])*

$$\partial_t u = \Delta u - Ru \quad \text{in } M \times (s, T] \quad (1.21)$$

*with*

$$\lim_{t \searrow s} \mathcal{Z}(x, t; y, s) = \delta_y(x). \quad (1.22)$$

*For any  $k \in \mathbb{Z}^+$ , let  $\mathcal{Z}_k = \mathcal{Z}_k(x, t; y, s)$ ,  $x, y \in M$ ,  $0 \leq s < t \leq T$ , be the Neumann Green function of the forward conjugate heat equation which satisfies*

$$\begin{cases} \partial_t \mathcal{Z}_k = \Delta \mathcal{Z}_k - R \mathcal{Z}_k & \text{in } B_k \times (s, T] \\ \frac{\partial \mathcal{Z}_k}{\partial \nu} = 0 & \text{on } \partial B_k \times (s, T] \\ \lim_{t \searrow s} \mathcal{Z}_k(x, t; y, s) = \delta_y(x) \end{cases} \quad (1.23)$$

*where  $\partial/\partial \nu$  is the derviative with respect to the unit outward normal on  $\partial B_k \times (s, T]$  and  $B_k = B_k(x_0)$  for some fix point  $x_0 \in M$ . Then  $\mathcal{Z}_k(x, t; y, s)$  converges uniformly on every compact subset of  $M \times (s, T]$  to  $\mathcal{Z}(x, t; y, s)$  as  $k \rightarrow \infty$ .*

*Proof.* By (1.20) and (1.23),

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B_k} \mathcal{Z}_k(x, t; y, s) dV_t(x) &= \int_{B_k} \Delta \mathcal{Z}_k(x, t; y, s) dV_t(x) = 0 \quad \forall 0 \leq s < t \leq T, k \in \mathbb{Z}^+ \\ \Rightarrow \int_{B_k} \mathcal{Z}_k(x, t; y, s) dV_t(x) &= \lim_{t' \searrow s} \int_{B_k} \mathcal{Z}_k(x, t'; y, s) dV_t(x) = 1 \quad \forall 0 \leq s < t \leq T, k \in \mathbb{Z}^+ \end{aligned} \quad (1.24)$$

Let  $R > 1$  and fix  $(y, s) \in M \times [0, T]$ . By (1.19) and (1.20), (1.7) and (1.9) holds with  $\alpha_3 = 2(n-1)k_0$ . Let  $G_k(x, t; y, s)$  be the Dirichlet Green function of (1.21) in  $M \times (0, T)$ . We now divide the proof into two cases.

**Case 1:**  $n \geq 3$

By (1.7), (1.9), (1.24), and Lemma 3.1 of [CTY], there exists a constant  $C_1 > 0$  such that for any  $s < t_1 < T$ ,  $k \geq 3R$ ,

$$\begin{aligned} \mathcal{Z}_k(x, t; y, s) &\leq \frac{C_1}{r_1^2 V_x(r_1)} \int_{t-4r_1^2}^t \int_{B_{2r_1}(x)} \mathcal{Z}_k(z, t; y, s) dV_0(z) dt \\ &\leq \frac{C'_1}{r_1^2 \min_{z \in B_R} V_z(r_1)} \int_{t-4r_1^2}^t \int_M \mathcal{Z}_k(z, t; y, s) dV_t(z) dt \\ &\leq \frac{4C'_1}{\min_{z \in B_R} V_z(r_1)} \quad \forall x \in \overline{B}_R, t_1 \leq t \leq T \end{aligned}$$

where  $r_1 = \min(1/2, \sqrt{t_1 - s}/4)$ . Hence the sequence  $\{\mathcal{Z}_k(\cdot, \cdot; y, s)\}$  are uniformly bounded on  $\overline{B}_R \times [t_1, T]$  for any  $s < t_1 < T$ ,  $k \geq 3R$ . By (1.23) and the parabolic Schauder estimates [LSU] the sequence  $\{\mathcal{Z}_k\}$  is uniformly bounded in  $C^{2,\beta}(\overline{B}_R \times [t_1, T])$  for some  $\beta \in (0, 1)$  for any  $s < t_1 < T$ ,  $k \geq 3R$ .

Let  $\{k_i\}_{i=1}^\infty$  be a sequence of positive integers such that  $k_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Then by the Ascoli Theorem and a diagonalization argument the sequence  $\{\mathcal{Z}_{k_i}\}_{i=1}^\infty$  has a subsequence which we may assume without loss of generality to be the sequence itself which converges uniformly on every compact subset of  $M \times (s, T]$  to some solution  $\tilde{\mathcal{Z}}(\cdot, \cdot; y, s)$  of (1.21) in  $M \times (s, T]$  as  $k \rightarrow \infty$ .

By the construction of  $\mathcal{Z}(x, t; y, s)$  in [CTY]  $G_k$  increases monotonically to  $\mathcal{Z}$  as  $k \rightarrow \infty$ . Let  $0 \leq s < t \leq T$ . By the maximum principle,

$$G_{k_i}(x, t; y, s) \leq \mathcal{Z}_{k_i}(x, t; y, s) \quad \forall x, y \in B_{k_i}, 0 \leq s < t \leq T, i \in \mathbb{Z}^+ \quad (1.25)$$

$$\Rightarrow \mathcal{Z}(x, t; y, s) \leq \tilde{\mathcal{Z}}(x, t; y, s) \quad \forall x, y \in M, 0 \leq s < t \leq T \quad \text{as } i \rightarrow \infty. \quad (1.26)$$

By (1.24) and (1.25),  $\forall 0 \leq s < t \leq T$ ,

$$\begin{aligned} \int_{B_R} G_{k_i}(x, t; y, s) dV_t(x) &\leq \int_{B_R} \mathcal{Z}_{k_i}(x, t; y, s) dV_t(x) \leq 1 \quad \forall k_i > R > 1 \\ \Rightarrow \int_{B_R} \mathcal{Z}(x, t; y, s) dV_t(x) &\leq \int_{B_R} \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) \leq 1 \quad \forall R > 1 \text{ as } i \rightarrow \infty \\ \Rightarrow \int_M \mathcal{Z}(x, t; y, s) dV_t(x) &\leq \int_M \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) \leq 1 \quad \text{as } R \rightarrow \infty \end{aligned} \quad (1.27)$$

By Lemma 5.1 of [CTY],

$$\int_M \mathcal{Z}(x, t; y, s) dV_t(x) = 1 \quad \forall y \in M, 0 \leq s < t \leq T. \quad (1.28)$$

By (1.26), (1.27), and (1.28),

$$\begin{aligned} \int_M \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) &= \int_M \mathcal{Z}(x, t; y, s) dV_t(x) = 1 \quad \forall y \in M, 0 \leq s < t \leq T \\ \Rightarrow \tilde{\mathcal{Z}}(x, t; y, s) &\equiv \mathcal{Z}(x, t; y, s) \quad \forall x, y \in M, 0 \leq s < t \leq T. \end{aligned}$$

Since the sequence  $\{k_i\}$  is arbitrary,  $\mathcal{Z}_k(x, t, y, s)$  converges to  $\mathcal{Z}(x, t, y, s)$  uniformly on every compact subset of  $M \times (s, T]$  as  $k \rightarrow \infty$ .

**Case 2:**  $n = 2$

Let  $h_0$  be the standard metric on the 2-sphere  $S^2$  with constant scalar curvature 1. Then  $(S^2, h(t))$ ,  $h = (h_{\alpha\beta})$ ,  $0 \leq t \leq T$ , with  $h(t) = (1+t)h_0$  is the solution of the backward Ricci flow on  $S^2$  (P.65 of [MT]). Consider the manifold  $\tilde{M} = M \times S^2$  with metric  $\tilde{g}$  given by  $\tilde{g}_{ij} = g_{ij}$ ,  $\tilde{g}_{\alpha\beta} = h_{\alpha\beta}$ ,  $\tilde{g}_{i\alpha} = \tilde{g}_{\alpha i} = 0$ . Then  $(\tilde{M}, \tilde{g})$  satisfies the backward Ricci flow on  $[0, T]$  with uniformly bounded Riemannian curvatures on  $[0, T]$ .

Hence as before there exist constants  $C_2 > 0$ ,  $C_3 > 0$ , such that

$$\begin{cases} \frac{1}{C_2} \tilde{g}(0) \leq \tilde{g}(t) \leq C_2 \tilde{g}(0) & \text{in } \tilde{M} \times [0, T] \\ \frac{1}{C_3} d\tilde{V} \leq d\tilde{V}_t \leq C_3 d\tilde{V} & \text{in } \tilde{M} \times [0, T] \end{cases} \quad (1.29)$$

where  $d\tilde{V}_t$  is the volume element of  $\tilde{M}$  with respect to the metric  $\tilde{g}(t)$  and  $d\tilde{V} = d\tilde{V}_0$ . For any  $x, y \in M$ ,  $x', y' \in S^2$ , let

$$\tilde{\mathcal{Z}}_k(x, x', t; y, y', s) = \mathcal{Z}_k(x, t; y, s) \quad \forall 0 \leq s < t \leq T.$$

Since  $\mathcal{Z}_k$  satisfies (1.21),

$$\partial_t \tilde{\mathcal{Z}}_k = \Delta_{\tilde{g}(t)} \tilde{\mathcal{Z}}_k - R \tilde{\mathcal{Z}}_k \quad \text{in } B_k \times S^2 \times (s, T). \quad (1.30)$$

Then by Lemma 3.1 of [CTY], (1.24), (1.29) and (1.30), there exists a constant  $C_4 > 0$  such that for any  $0 \leq s < t_1 \leq T$ ,  $k \geq 3R$ ,

$$\begin{aligned} \tilde{\mathcal{Z}}_k(x, x', t; y, y', s) &\leq \frac{C_4}{r_1^2 \tilde{V}_{(x, x')}(r_1)} \int_{t-4r_1^2}^t \int_{\tilde{B}_{2r_1}(x, x')} \tilde{\mathcal{Z}}_k(z, t; y, y', s) d\tilde{V}(z) dt \\ &\leq \frac{C'_4}{r_1^2 \tilde{V}_{(x, x')}(r_1)} \int_{t-4r_1^2}^t \int_{B_k} \mathcal{Z}_k(w, t; y, s) dV_t(w) dt \\ &\leq \frac{4C'_4}{\tilde{V}_{(x, x')}(r_1)} \quad \forall x, y \in \overline{B}_R, x', y' \in S^2, t_1 \leq t \leq T \\ \Rightarrow \mathcal{Z}_k(x, t; y, s) &\leq \frac{4C'_4}{\min_{\substack{w \in \overline{B}_R \\ x' \in S^2}} \tilde{V}_{(w, x')}(r_1)} \quad \forall x \in B_R, t_1 \leq t \leq T \end{aligned} \quad (1.31)$$



where  $r_1 = \min(1/2, \sqrt{t_1 - s}/4)$ ,  $\tilde{B}_{2r_1}(x, x')$  is the geodesic ball of radius  $2r_1$  and center  $(x, x')$  in  $\tilde{M}$  with respect to the metric  $\tilde{g}(0)$  and  $\tilde{V}_{(x, x')}(r_1)$  is the volume of  $\tilde{B}_{r_1}(x, x')$  with respect to the metric  $\tilde{g}(0)$ . Hence the sequence  $\{\tilde{Z}_k(\cdot, \cdot; y, s)\}_{k=1}^\infty$  are uniformly bounded on  $\overline{B}_R \times [t_1, T]$  for any  $s < t_1 \leq T$ ,  $k \geq 3R$ .

By a similar argument the sequence  $\{G_k(\cdot, \cdot; y, s)\}_{k=1}^\infty$  are uniformly bounded on  $\overline{B}_R \times [t_1, T]$  for any  $s < t_1 \leq T$ ,  $k \geq 3R$ . Then by the same argument as in [CTY],  $G_k$  increases monotonically to  $\mathcal{Z}$  as  $k \rightarrow \infty$  and  $\mathcal{Z}(x, t; y, s)$  satisfies (1.28). By (1.28), (1.31), and an argument similar to case 1,  $\mathcal{Z}_k(x, t; y, s)$  converges to  $\mathcal{Z}(x, t; y, s)$  uniformly on every compact subset of  $M \times (s, T]$  as  $k \rightarrow \infty$  and the theorem follows.

**Corollary 1.5.** *Let  $(M, g(t))$ ,  $0 \leq t \leq T$ , and  $\mathcal{Z}(x, t; y, s)$ ,  $x, y \in M$ ,  $0 \leq s < t \leq T$ , be as in Theorem 1.4. Suppose  $\tilde{\mathcal{Z}}(x, t; y, s)$  is a fundamental solution of the forward conjugate heat equation which satisfies (1.21), (1.22) and*

$$\forall y \in M, \max_{s \leq t \leq T} \int_{B_R} \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) \leq o(R) \quad \forall 0 \leq s < T \quad \text{as } R \rightarrow \infty, \quad (1.32)$$

then

$$\tilde{\mathcal{Z}}(x, t; y, s) \equiv \mathcal{Z}(x, t; y, s) \quad \forall x, y \in M, 0 \leq s < t \leq T. \quad (1.33)$$

*Proof.* By (1.32) and an argument similar to the proof of Lemma 5.1 of [CTY],

$$\int_M \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) = 1 \quad \forall y \in M, 0 \leq s < t \leq T. \quad (1.34)$$

Let  $G_k$  be as in the proof of Theorem 1.4. By the maximum principle,

$$G_k(x, t; y, s) \leq \tilde{\mathcal{Z}}(x, t; y, s) \quad \forall x, y \in B_k, 0 \leq s < t \leq T, k \in \mathbb{Z}^+. \quad (1.35)$$

Since  $\mathcal{Z}$  satisfies (1.28), by (1.34), (1.35) and an argument similar to the proof of Theorem 1.4 the corollary follows.

**Theorem 1.6.** *Let  $(M, g(t))$ ,  $0 \leq t \leq T$ , and  $\mathcal{Z}(x, t; y, s)$ ,  $x, y \in M$ ,  $0 \leq s < t \leq T$ , be as in Theorem 1.4. Let  $\tilde{\mathcal{Z}}(x, t; y, s)$  be a fundamental solution of the forward conjugate heat equation which satisfies (1.21) and (1.22). Then (1.33) holds if and only if there exist constants  $C > 0$  and  $D > 0$  such that*

$$\tilde{\mathcal{Z}}(x, t; y, s) \leq \frac{C}{V_y(\sqrt{t-s})} e^{-\frac{r^2(x, y)}{D(t-s)}} \quad \forall 0 \leq s < t \leq T. \quad (1.36)$$

*Proof.* The case (1.33) implies (1.36) was proved in [CTY]. Hence we only need to show that (1.36) implies (1.33). Suppose there exist constants  $C > 0$  and  $D > 0$  such that

(1.36) holds. By the proof of Theorem 1.4 (1.9) holds for some constant  $\alpha_3 > 0$ . Then by (1.9) and (1.36),

$$\begin{aligned} \int_M \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) &\leq \frac{C}{V_y(\sqrt{t-s})} \int_M e^{-\frac{r^2(x,y)}{D(t-s)}} dV_t(x) \\ &\leq \frac{C}{V_y(\sqrt{t-s})} \int_0^\infty e^{-\frac{r^2}{D(t-s)}} dV_y(r) \\ &= C \int_0^\infty \frac{V_y(r)}{V_y(\sqrt{t-s})} e^{-\frac{r^2}{D(t-s)}} d\left(\frac{r^2}{D(t-s)}\right). \end{aligned} \quad (1.37)$$

Let  $V_{k_0}(r)$  be the volume of the geodesic ball of radius  $r$  in the space form with constant sectional curvature  $-k_0$ . Let  $\delta = \sqrt{t-s}$  and  $a = (r/\sqrt{t-s}) + 1$ . By (1.19) and the volume comparison theorem [C],

$$\begin{aligned} \frac{V_y(r)}{V_y(\sqrt{t-s})} &\leq \frac{V_y(r + \sqrt{t-s})}{V_y(\sqrt{t-s})} \leq \frac{V_{k_0}(r + \sqrt{t-s})}{V_{k_0}(\sqrt{t-s})} = \frac{V_{k_0}(a\delta)}{V_{k_0}(\delta)} = a^n \frac{V_{a^2 k_0}(\delta)}{V_{k_0}(\delta)} \\ &\leq a^n \frac{V_{a^2 k_0}(\sqrt{T})}{V_{k_0}(\sqrt{T})} = \frac{V_{k_0}(a\sqrt{T})}{V_{k_0}(\sqrt{T})}. \end{aligned} \quad (1.38)$$

Now

$$\begin{aligned} V_{k_0}(a\sqrt{T}) &= \int_0^{a\sqrt{T}} \left( \frac{1}{\sqrt{k_0}} \sinh(\sqrt{k_0}\rho) \right)^{n-1} d\rho \leq C e^{(n-1)\sqrt{k_0 T} a} \\ &= C e^{(n-1)\sqrt{k_0 T} (\frac{r}{\sqrt{t-s}} + 1)} \\ &\leq C e^{C' \frac{r}{\sqrt{t-s}}}. \end{aligned} \quad (1.39)$$

By (1.37), (1.38), and (1.39),

$$\int_M \tilde{\mathcal{Z}}(x, t; y, s) dV_t(x) \leq C \int_0^\infty e^{-\frac{r^2}{D(t-s)} + C' \frac{r}{\sqrt{t-s}}} d\left(\frac{r^2}{D(t-s)}\right) = C_T < \infty \quad \forall 0 \leq s < t \leq T$$

for some constant  $C_T > 0$  depending on  $k_0$  and  $T$ . Hence by Corollary 1.5 (1.33) holds.

## Section 2

In this section we will give a detail proof of the convergence of fundamental solutions of conjugate heat equation for Ricci flow which was used without proof by Perelman in his proof of the pseudolocality theorem for Ricci flow [P].

**Theorem 2.1.** *Let  $\alpha > 0$  and let  $(M_k \times (-\alpha, 0], x_k, g_k(t))$  be a sequence of pointed Ricci flow (0.1) where each  $M_k$  is either a closed manifold or a non-compact manifold with bounded curvature such that  $(M_k, g_k(t))$  is complete for each  $-\alpha < t \leq 0$ . Suppose*

$$|Rm_k|(x, t) \leq C_1 \quad \forall x \in B_k(x_k, A_k), -\alpha < t \leq 0, k \in \mathbb{Z}^+ \quad (2.1)$$

for some constant  $C_1 > 0$  and sequence  $\{A_k\}$ ,  $A_k \rightarrow \infty$  as  $k \rightarrow \infty$ , and

$$(M_k \times (-\alpha, 0], x_k, g_k(t))$$

converges in the  $C^\infty$ -sense to some pointed Ricci flow  $(M \times (-\alpha, 0], x_\infty, g_\infty)$  as  $k \rightarrow \infty$  where  $B_k(x_k, A_k) = B_{g_k(0)}(x_k, A_k)$ . That is there exists an exhausting sequence  $U_1 \subset U_2 \subset \dots \subset M$  of open sets each containing  $x_\infty$  and each with compact closure in  $M$  and diffeomorphisms  $\Phi_k$  of  $U_k$  to open sets  $V_k$  of  $M_k$  such that  $\Phi_k(x_\infty) = x_k \quad \forall k \in \mathbb{Z}^+$  and the pull-back metric  $\Phi_k^*(g_k)$  converges uniformly to  $g_\infty$  on every compact subset of  $M \times (-\alpha, 0]$  as  $k \rightarrow \infty$ .

If  $u_k$  satisfies the conjugate heat equation,

$$u_t + \Delta_k u - R_{g_k(t)} u = 0 \quad (2.2)$$

in  $M_k \times (-\alpha, 0)$  with

$$\lim_{t \nearrow 0} u(x, t) = \delta_{x_k}$$

where  $\Delta_k = \Delta_{g_k(t)}$ , then  $\Phi_k^*(u_k)$  will converge uniformly on every compact subset of  $M \times (-\alpha, 0)$  to the minimal fundamental solution  $u$  of the conjugate heat equation

$$u_t + \Delta_{g_\infty(t)} u - R_{g_\infty(t)} u = 0 \quad (2.3)$$

of  $(M, g_\infty)$  in  $M \times (-\alpha, 0)$  with

$$\lim_{t \nearrow 0} u(x, t) = \delta_{x_\infty} \quad (2.4)$$

as  $k \rightarrow \infty$ .

*Proof.* For simplicity we will write  $B_r(x)$ ,  $V_k$ ,  $V$ ,  $dV_k^t$ ,  $dV_t$ , for  $B_{g_\infty(0)}(x, r)$ ,  $\text{Vol}_{g_k(0)}$ ,  $\text{Vol}_{g_\infty(0)}$ ,  $dV_{g_k(t)}$ , and  $dV_{g_\infty(t)}$  respectively. We also let  $dV_k = dV_k^0$ . Note that  $u_k > 0$  in  $M_k \times (-\alpha, 0)$  and ([CTY])

$$\int_{M_k} u_k(y, t) dV_k^t(y) = 1 \quad \forall -\alpha < t \leq 0, k \in \mathbb{Z}^+. \quad (2.5)$$

Since  $g_k$  satisfies (0.1) in  $M_k \times (-\alpha, 0]$ , by (2.1) there exists a constant  $C_2 > 1$  such that

$$\begin{cases} \frac{1}{C_2} g_k(x, s) \leq g_k(x, t) \leq C_2 g_k(x, s) & \forall x \in B_k(x_k, A_k), -\alpha < s, t \leq 0, k \in \mathbb{Z}^+ \\ \frac{1}{C_2} dV_k^s(x) \leq dV_k^t(x) \leq C_2 dV_k^s(x) & \forall x \in B_k(x_k, A_k), -\alpha < s, t \leq 0, k \in \mathbb{Z}^+. \end{cases} \quad (2.6)$$

Let  $r_k(x, y, t)$  be the geodesic distance between  $x, y \in M_k$  with respect to the metric  $g_k(t)$  and let  $r_k(x, y) = r_k(x, y, 0)$ . Let  $r(x, y)$  be the geodesic distance between  $x, y \in M$  with respect to the metric  $g_\infty(0)$ . Let  $R > 1$ ,  $-\alpha < t_1 < t_2 < 0$ , and let  $r_1 = \min(1/2, \sqrt{-t_2}/4)$ .

We choose  $k'_1 \in \mathbb{Z}^+$  such that  $\overline{B_{R+2}(x_\infty)} \subset U_{k_1}$  and  $A_k \geq 6\sqrt{C_2}R$  for all  $k \geq k'_1$ . Then by (2.6),

$$\frac{1}{\sqrt{C_2}}r_k(x, y, s) \leq r_k(x, y, t) \leq \sqrt{C_2}r_k(x, y, s) \quad \forall x, y \in B_k(x_k, 2R), -\alpha < s, t \leq 0, k \geq k'_1. \quad (2.7)$$

Since  $\overline{B_R(x_\infty)} \times [t_1, t_2]$  is compact, there exist

$$z_1, z_2, \dots, z_m \in \overline{B_R(x_\infty)}, s_1, s_2, \dots, s_m \in [t_1, t_2]$$

such that

$$\overline{B_R(x_\infty)} \times [t_1, t_2] \subset \cup_{j=1}^m B_{\frac{r_1}{2}}(z_j) \times [s_j, s_j + r_1^2]. \quad (2.8)$$

Let  $z_j^k = \Phi_k(z_j)$ . Since  $\Phi_k^*(g_k)$  converges uniformly to  $g_\infty$  on  $\overline{B_{2R}(x_\infty)} \times [t_1, 0]$  as  $k \rightarrow \infty$ , there exists  $k'_2 \geq k'_1$  such that for any  $k \geq k'_2$ ,  $j = 1, \dots, m$ ,

$$\begin{cases} \Phi_k(\overline{B_R(x_\infty)}) \subset \overline{B_k(x_k, R + (1/2))} \\ \Phi_k(\overline{B_{r_1/\sqrt{2C_2}}(z_j)}) \subset \overline{B_k(z_j^k, r_1/\sqrt{C_2})} \\ \Phi_k(\overline{B_{\frac{r_1}{2}}(z_j)}) \subset \overline{B_k(z_j^k, r_1)}. \end{cases} \quad (2.9)$$

By (2.9),

$$z_j^k \in \overline{B_k(x_k, R + (1/2))} \quad \forall j = 1, \dots, m, k \geq k'_2 \quad (2.10)$$

By (2.7) and (2.10),

$$B_k^s(z_j^k, C_2^{-\frac{1}{2}}r) \subset B_k^t(z_j^k, r) \quad \forall t_1 \leq s, t \leq 0, k \geq k'_2, j = 1, \dots, m, 0 < r \leq r_1. \quad (2.11)$$

Let  $\{k_i\}_{i=1}^\infty \subset \mathbb{Z}^+$  be a sequence such that  $k_i \rightarrow \infty$  as  $i \rightarrow \infty$ . We now divide the proof into two cases.

**Case 1:**  $n \geq 3$

By (2.1), (2.2), (2.5), (2.6), (2.7), (2.9), (2.10), (2.11) and an argument similar to the proof of Lemma 3.1 of [CTY] and Theorem 3.1 of [KZ] and there exists a constant  $C_3 > 0$  such that for any  $x \in \overline{B_k(z_j^k, r_1)}$ ,  $s_j \leq t \leq s_j + r_1^2$ ,  $k \geq k'_2$ ,  $j = 1, \dots, m$ ,

$$\begin{aligned} u_k(x, t) &\leq \frac{C_3}{r_1^2 V_k(B_k(z_j^k, C_2^{-\frac{1}{2}}r_1))} \int_{s_j}^{s_j + 4r_1^2} \int_{B_k(z_j^k, 2r_1)} u_k(y, t) dV_k(y) dt \\ &\leq \frac{C_2 C_3}{r_1^2 V_k(B_k(z_j^k, C_2^{-\frac{1}{2}}r_1))} \int_{s_j}^{s_j + 4r_1^2} \int_{B_k(z_j^k, 2r_1)} u_k(x, t) dV_k^t(y) dt \\ &\leq \frac{4C_2 C_3}{V_{\Phi_k^*(g_k(0))}(B_{r_1/\sqrt{2C_2}}(z_j))}. \end{aligned} \quad (2.12)$$

Since  $\Phi_k^*(g_k)$  converges uniformly to  $g_\infty$  on  $\overline{B_{2R}(x_\infty)} \times [t_1, 0]$  as  $k \rightarrow \infty$ , there exist  $k'_3 \geq k'_2$  and a constant  $C_4 > 0$  such that

$$V_{\Phi_k^*(g_k(0))}(B_{r_1/\sqrt{2C_2}}(z_j)) \geq C_4 \quad \forall k \geq k'_3, j = 1, \dots, m. \quad (2.13)$$

By (2.8), (2.9), (2.12) and (2.13),

$$\Phi_k^*(u_k)(y, t) \leq \frac{4C_2C_3}{C_4} \quad \forall y \in \overline{B_R(x_\infty)}, t_1 \leq t \leq t_2, k \geq k'_3. \quad (2.14)$$

Hence the sequence  $\{\Phi_k^*(u_k)\}_{k=1}^\infty$  are uniformly bounded on  $\overline{B_R(x_\infty)} \times [t_1, t_2]$  for any  $-\alpha < t_1 < t_2 < 0$ . Since  $\Phi_k^*(u_k)$  satisfies the conjugate heat equation on  $\overline{B_R(x_\infty)} \times (-\alpha, 0)$ , by (2.2) and the injectivity radius estimates of [CLY] and the uniform convergence of  $\Phi_k^*(g_k)$  to  $g_\infty$  on every compact subset of  $M \times (-\alpha, 0]$  as  $k \rightarrow \infty$ , one can apply the parabolic Schauder estimates of [LSU] to conclude that for any  $R > 1$  and  $-\alpha < t_1 < t_2 < 0$   $\{\Phi_k^*(u_k)\}_{k=1}^\infty$  are uniformly bounded in  $C^{2,\beta}(\overline{B_R(x_\infty)} \times [t_1, t_2])$  for some  $\beta \in (0, 1)$ .

**Case 2:**  $n = 2$

By considering  $\widetilde{M} = M \times S^2$  and using an argument similar to the proof of Theorem 1.4 and case 1 one can also conclude that when  $n = 2$ , for any  $R > 1$  and  $-\alpha < t_1 < t_2 < 0$   $\{\Phi_k^*(u_k)\}_{k=1}^\infty$  are uniformly bounded in  $C^{2,\beta}(\overline{B_R(x_\infty)} \times [t_1, t_2])$  for some  $\beta \in (0, 1)$ .

Hence by case 1 and case 2, the Ascoli Theorem and a diagonalization argument  $\{\Phi_{k_i}^*(u_{k_i})\}_{i=1}^\infty$  has a subsequence which we may assume without loss of generality to be the sequence  $\{\Phi_{k_i}^*(u_{k_i})\}_{i=1}^\infty$  itself that converges uniformly on every compact subset of  $M \times (-\alpha, 0)$  to a solution  $u$  of the conjugate heat equation of  $(M, g_\infty)$  in  $M \times (-\alpha, 0)$  as  $i \rightarrow \infty$ .

By (2.1), (2.6), (2.7), and an argument similar to the proof of Theorem 5.1 and (5.2) of [CTY], there exist constants  $C > 0$  and  $D > 0$  such that

$$\begin{aligned} u_k(x, t) &\leq \frac{C}{V_k(B_k(x_k, \sqrt{t}))} e^{-\frac{r_k(x, x_k)^2}{D|t|}} \quad \forall x \in B_k(x_k, R), -\alpha < t < 0, k \geq k'_3 \\ \Rightarrow u(x, t) &\leq \frac{C}{V(B_{\sqrt{t}}(x_\infty))} e^{-\frac{r(x, x_\infty)^2}{D|t|}} \quad \forall x \in M, -\alpha < t < 0 \quad \text{as } k = k_i \rightarrow \infty. \end{aligned} \quad (2.15)$$

Let  $\psi \in C_0^\infty(M)$ . Then  $\text{supp } \psi \subset B_{R_1}(x_\infty)$  for some constant  $R_1 > 0$ . Choose  $k'_4 \geq k'_3$  such that  $\overline{B_{R_1}(x_\infty)} \subset U_k$  for all  $k \geq k'_4$ . Let

$$\psi_k(x) = \Phi_k^*(\psi)(x) = \begin{cases} \psi(\Phi_k^{-1}(x)) & \text{if } x \in V_k \\ 0 & \text{if } x \notin V_k. \end{cases}$$

Then  $\psi_k \in C_0^\infty(M_k)$  for all  $k \geq k'_4$  and  $\psi_k(x_k) = \psi(\Phi_k^{-1}(x_k)) = \psi(x_\infty)$ . Let  $t_1 \in (-\alpha, 0)$ .

Then by (0.1), (2.2) and (2.5),  $\forall t_1 \leq t < 0, k \geq k'_4$ ,

$$\begin{aligned}
\left| \int_M \Phi_k^*(u_k) \psi dV_{\Phi_k^*(g_k(t))} - \psi(x_\infty) \right| &= \left| \int_{M_k} u_k \psi_k dV_k^t - \psi_k(x_k) \right| \\
&= \left| \int_0^t \int_{M_k} \psi_k \left( \frac{\partial u_k}{\partial t} - R_{g_k(t)} u_k \right) dV_k^t dt \right| \\
&= \left| \int_0^t \int_{M_k} \psi_k \Delta_k u_k dV_k^t dt \right| \\
&= \left| \int_0^t \int_{M_k} u_k \Delta_k \psi_k dV_k^t dt \right| \\
&\leq \max |\Delta_k \psi_k| \left( \int_{M_k} u_k dV_k^t \right) |t| \\
&\leq |t| \frac{\max}{B_{R_1}(x_\infty) \times [t_1, 0]} |\Delta_{\Phi_k^*(g_k(t))} \psi|. \tag{2.16}
\end{aligned}$$

Letting  $k \rightarrow \infty$  in (2.16),

$$\begin{aligned}
\left| \int_M u \psi dV(t) - \psi(x_\infty) \right| &\leq |t| \frac{\max}{B_{R_1}(x_\infty) \times [t_1, 0]} |\Delta_{g_\infty(t)} \psi| \quad \forall t_1 \leq t < 0 \\
\Rightarrow \lim_{t \nearrow 0} \int_M u \psi dV(t) &= \psi(x_\infty) \quad \text{as } t \rightarrow 0.
\end{aligned}$$

Hence  $u$  satisfies (2.3) in  $M \times (-\alpha, 0)$  and (2.4) holds. By (2.15) and Theorem 1.6  $u$  is the unique minimal fundamental solution of the conjugate heat equation (2.3) in  $M \times (-\alpha, 0)$  which satisfies (2.4). Since the sequence  $\{k_i\}_{i=1}^\infty$  is arbitrary,  $\Phi_k^*(u_k)$  converges uniformly on every compact subset of  $M \times (-\alpha, 0)$  to the minimal fundamental solution of the conjugate heat equation of  $(M, g_\infty)$  in  $M \times (-\alpha, 0)$  which satisfies (2.4) as  $k \rightarrow \infty$  and the theorem follows.

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